

# MEDIAN AND MEAN OF THE SUPREMUM OF $L^2$ NORMALIZED RANDOM HOLMORPHIC FIELDS

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**ABSTRACT.** We prove that the expected value and median of the supremum of  $L^2$  normalized random holomorphic fields of degree  $n$  on  $m$ -dimensional Kähler manifolds are asymptotically of order  $\sqrt{m \log n}$ . There is an exponential concentration of measure of the sup norm around this median value. Prior results only gave the upper bound. The estimates are based on the entropy methods of Dudley and Sudakov combined with a precise analysis of the relevant distance functions and covering numbers using off-diagonal asymptotics of Bergman kernels. Recent work on the value distribution are also used.

The purpose of this note is to determine the asymptotic mean and median of the sup-norm functionals

$$\mathcal{L}_\infty^n : H^0(M, L^n) \rightarrow \mathbb{R}_+, \quad \mathcal{L}_\infty^n(s_n) = \sup_{z \in M} |s_n(z)|_{h^n}$$

on the subspace

$$(1) \quad SH^0(M, L^n) = \{s_n \in H^0(M, L^n) : \|s_n\|_{h^n}^2 := \int_M |s_n(z)|_{h^n}^2 dV = 1\}$$

of  $L^2$  normalized random holomorphic sections of the  $n$ th power  $(L^n, h^n) \rightarrow (M, \omega)$  of a positive Hermitian holomorphic line bundle over a compact Kähler manifold. As discussed at length in [SZ, SZ2] and [FZ] (among many other articles), holomorphic sections of positive line bundles are the analogues on compact complex manifold of polynomials of degree  $n$ , and in the special case of  $M = \mathbb{CP}^m$  and  $L = \mathcal{O}(1)$ ,  $H^0(M, L^n)$  is the space of homogeneous holomorphic polynomials of degree  $n$  on  $\mathbb{C}^{m+1}$  (see §1 for background). The inner product  $\|s_n\|_{h^n}^2$  induces a unit mass spherical Haar measure  $\nu_n$  on  $SH^0(M, L^n)$  and we are interested in the statistical properties of  $\mathcal{L}_\infty^n$  in this spherical ensemble. As discussed in [FZ, SZ], we regard the spherical ensemble as primary since our goal is to measure sup norms of  $L^2$  normalized sections. We denote the expectation of a random variable in any measure  $\mu$  by  $\mathbb{E}_\mu$ , and the median by  $\mathcal{M}_\mu$ .

**THEOREM 1.** *The median and mean value of  $\mathcal{L}_\infty^n$  on  $(SH^0(M, L^n), \nu_n)$  satisfy*

$$\mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n) = \sqrt{m \log n} + o(\sqrt{\log n}), \quad \text{resp.} \quad \mathbb{E}_{\nu_n} \mathcal{L}_\infty^n = \sqrt{m \log n} + o(\sqrt{\log n}).$$

The upper bound on the median with an unspecified constant was proved in [SZ], and the question of finding its true order of magnitude was raised there. To

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*Date:* March 20, 2013.

Research partially supported by NSF grant DMS-1206527.

our knowledge, the lower bound is new, although there are many prior results on suprema of random processes in other contexts, and the proof is based in part on classical entropy methods of Dudley [D, D2] and Sudakov [S]. The main new ingredient is the analysis of the pseudo-metrics  $\mathbf{d}_n$  induced by the holomorphic random fields, which makes use of the off-diagonal asymptotics of the Szegő kernel in the complex geometric setting of [SZ2]. Use is also made of recent results on the value-distribution of the fields [FZ] in §4 in getting precise bounds. The same methods apply in the real domain to random spherical harmonics and their generalization to random Riemannian waves, except that the pseudo-metric in that setting is determined by the spectral projections kernels for the Laplacian.

The precise value of the median is needed to obtain a concrete Levy concentration of measure result for the sup norm. Levy concentration (see (50)) states that a Lipschitz functional is exponentially concentrated around its median value [Le]. As noted in [SZ], the functional  $\mathcal{L}_\infty^n$  is Lipschitz with the estimate (51) of Levy concentration of measure theorem, Theorem 1 thus has the following

**COROLLARY 1.** *There exists constants  $C, c > 0$  independent of  $n$  so that*

$$\nu_n\{s_n \in SH^0(M, L^n) : |\mathcal{L}_\infty^n(s_n) - \sqrt{m \log n}| \geq \epsilon \sqrt{\log n}\} \leq Cn^{-c\epsilon^2}.$$

for any  $\epsilon > 0$ .

Throughout, we use  $c$  and  $C$  to denote positive constants which may differ in each instance. The symbol  $A \sim B$  means  $A$  and  $B$  are bounded from above and below by positive constants independent of  $n$ .

**0.1. Sketch of the proof.** The spherical measures  $\nu_n$  are asymptotically equivalent to certain Gaussian measure which we call *normalized Gaussian measure* on  $H^0(M, L^n)$  (see (13)-(14)). We first study the expectation and median of  $\mathcal{L}_\infty^n$  of normalized Gaussian random sections in Theorem 3.1; we then derive the result for  $\nu_n$ .

Let us recall the entropy estimates on suprema of Gaussian (or sub-Gaussian) random processes. Let  $(M, d)$  be a compact metric space. Given a centered random process  $\{Y_x : x \in M\}$  (i.e.  $\mathbb{E}Y_x = 0$  for all  $x \in M$ ), a pseudometric on  $M$  may be defined by

$$(2) \quad \mathbf{d}(x, y) = \sqrt{\mathbb{E}|Y_x - Y_y|^2}.$$

The process  $\{Y_x : x \in M\}$  is called *sub-Gaussian* if

$$(3) \quad \forall x, y \in M, \forall t > 0, \quad \mathbb{P}[|Y_x - Y_y| \geq t] \leq 2 \exp \left[ -\frac{b t^2}{\mathbf{d}^2(x, y)} \right]$$

for some constant  $b > 0$ . A Gaussian processes is sub-Gaussian.

Entropy estimates for suprema of (sub-) Gaussian processes involve the  $\epsilon$ -covering number  $N(M, \mathbf{d}, \epsilon)$  of  $(M, \mathbf{d})$ , i.e. the minimal cardinality of an  $\epsilon$ -dense subset of  $M$ , i.e.

$$N(M, \mathbf{d}, \epsilon) := \inf\{\#\mathcal{N} : \mathcal{N} \subset M : \forall x \in M \exists y \in \mathcal{N} : \mathbf{d}(x, y) \leq \epsilon\}.$$

Dudley's entropy upper bound state:

**THEOREM 2.** [D, D2, Ka, MP, Li] *Let  $\{Y_x : x \in M\}$  be a centered (sub-)Gaussian random process. Then*

$$(4) \quad \mathbb{E} \sup_{x \in M} |Y_x| \leq C \int_0^\infty \sqrt{\log N(M, \mathbf{d}, \varepsilon)} \, d\varepsilon,$$

where  $C > 0$  depend only on the constant  $b$  in the (sub-)Gaussian estimate for the process.

If  $Y_x$  is a centered Gaussian process, Sudakov's minoration gives the lower bound,

**THEOREM 3.** [S, Li]

$$(5) \quad \mathbb{E} \sup_{x \in M} |Y_x| \geq c\epsilon \sqrt{\log N(M, \mathbf{d}, \epsilon)}$$

In Lemmas 2.1 and 2.2, we relate the distance  $\mathbf{d}_n$  for the  $n$ th normalized Gaussian process to the distance between points of the Kodaira (or coherent states) embeddings

$$(6) \quad \Phi_n : M \rightarrow SH^0(M, L^n).$$

As proved in [Ti, Ze], this embedding is an asymptotically isometric embedding when properly normalized. Hence on very small length scales, the intrinsic Kähler distance and the extrinsic  $L^2$  distance of  $H^0(M, L^n)$  when restricted to  $\Phi_n(M)$  are very similar. The precise comparison is in Lemma 2.4.

We first prove the upper and lower bounds on expected sup norms for the *normalized Gaussian ensemble* in Section 3 with unspecified constants; the case of the spherical ensemble follows Lemma 1.1. Bounds of the median then follow by the Levy concentration theorem (§3.2). In the last section §4, we use the exact asymptotic formula in [FZ] for the value distribution to estimate the constants in upper and lower bounds and show that the mean and the median are asymptotic to the sharp bound  $\sqrt{m \log n}$ .

**0.2. Prior results.** The study of sup-norms of Gaussian random fields has a long history, and we only indicate a few of the classical results. In [SaZy], Salem-Zygmund studied sup norms

$$M_n(t) := \max_x |P_n(x, t)|$$

of random trigonometrical polynomials of the form,

$$P_n(x, t) = \sum_{m=0}^n r_m \phi_m(t) \cos(mx),$$

where  $\{\phi_m\}$  is the orthonormal basis of Rademacher functions and  $R_n = \sum_{m=1}^n r_m^2$ . We recall that the Rademacher system is the orthogonal system  $\{\phi_m(t) := \text{sgn}(\sin 2^m \pi t)\}_{m=1}^\infty$  for  $t \in [0, 1]$ . Let  $M_n(t) = \max_x |P_n(x, t)|$ . In Theorems 4.3.1 resp. 4.5.1 of [SaZy], it is proved that

$$c(\gamma) \leq \liminf_{n \rightarrow \infty} \frac{M_n(t)}{(R_n \log n)^{\frac{1}{2}}} \leq \limsup_{n \rightarrow \infty} \frac{M_n(t)}{(R_n \log n)^{\frac{1}{2}}} \leq A \quad \text{almost surely.}$$

They also proved,

$$\mathbb{P}\{M_n(t) < C(R_n \log n)^{\frac{1}{2}}\} \rightarrow 1$$

as  $C$  or  $n$  large enough.

Kahane [Ka] gives an upper bound for sup-norms of Gaussian random functions of the form

$$P(t_1, \dots, t_m) = \sum a_j f_j(t_1, \dots, t_m)$$

where  $\{f_j\}$  are complex trigonometric polynomials in  $m$  variables of degrees less than or equal to  $n$ ,  $a_j$  are normal random variables and  $\sum$  is a finite sum. In [Ka], (Chapter 6, Theorem 3), it is proved that

$$\mathbb{P} \{ \|P\|_\infty \geq C(m \sum \|f_j\|_\infty^2 \log n)^{\frac{1}{2}} \} \leq n^{-2} e^{-m}.$$

In [SZ] upper bounds on the expected value of  $\mathcal{L}_\infty^n$  in the spherical ensembles of this article are proved:

$$\nu_n \left\{ s_n \in SH^0(M, L^n) : \sup_M |s_n|_{h^n} > c\sqrt{\log n} \right\} < O\left(\frac{1}{n^2}\right),$$

for some constant  $c < +\infty$ . (In fact, for any  $k > 0$ , the probabilities are of order  $O(n^{-k})$  if one chooses  $c$  to be sufficiently large.) It is also proved that sequences of sections  $s_n \in SH^0(M, L^n)$  satisfy:

$$\|s_n\|_\infty = O(\sqrt{\log n}) \quad \text{almost surely.}$$

The proof is based on the same ingredients as the Dudley entropy bound of this note.

## 1. BACKGROUND

In this section, we go over the geometric background to our setting and the facts about Szegő kernels that we need in the proofs of the main results.

**1.1. Kähler geometry.** The setting consists of a compact Kähler manifold  $(M, \omega)$  of complex dimension  $m$  and a positive Hermitian holomorphic line bundle  $(L, h) \rightarrow M$ . We fix a local non-vanishing holomorphic section  $e$  of  $L$  over an open set  $U \subset M$  such that locally  $L|_U \cong U \times \mathbb{C}$ . We define the Kähler potential of  $\omega$  by  $|e|_h = h(e, e)^{1/2} = e^{-\phi}$ . The curvature of the Hermitian metric  $h$ ,

$$(7) \quad \Theta_h = -\partial\bar{\partial} \log |e|_h^2$$

is a positive  $(1, 1)$  form and  $\omega = \frac{i}{2} \Theta_h$  [GH].

The Hermitian metric  $h$  induces a Hermitian metric  $h^n$  on the  $n$ th tensor power  $L^n = L \otimes \dots \otimes L$  of  $L$ , given by  $|e^{\otimes n}|_{h^n} = |e|_h^n$ . In local coordinate, we can write a global holomorphic section as  $s_n = f_n e^{\otimes n}$  where  $f_n$  is a holomorphic function on  $U$ , and  $|s_n|_{h^n} = |f_n| e^{-\frac{n\phi}{2}}$ .

The spaces  $H^0(M, L^n)$  of global holomorphic sections of  $L^n$  provide generalizations of polynomials of degree  $n$  to  $M$ . By the Riemann-Roch formula, the dimension  $d_n = \dim H^0(M, L^n)$  grows at the rate

$$(8) \quad d_n = \frac{c_1(L)^m}{m!} n^m + O(n^{m-1}).$$

We define an inner product on  $H^0(M, L^n)$  by

$$(9) \quad \langle s_1^n, s_2^n \rangle_{h^n} = \int_M h^n(s_1^n, s_2^n) dV, \quad s_1^n, s_2^n \in H^0(M, L^n)$$

where  $dV = \frac{\omega^m}{m!}$  is the volume form. We assume the volume is normalized as  $\int_M dV = 1$ . In local coordinates,

$$(10) \quad \langle s_1^n, s_2^n \rangle_{h^n} = \int_M f_1 \bar{f}_2 e^{-n\phi} dV$$

where we write  $s_1^n = f_1^n e^{\otimes n}$  and  $s_2^n = f_2^n e^{\otimes n}$  in the local coordinate.

We choose an orthonormal basis  $\{s_j^n\}$  under this inner product and we can write every element in  $H^0(M, L^n)$  as the orthogonal series

$$(11) \quad s_n = \sum_{j=1}^{d_n} a_j s_j^n,$$

where  $s_j^n = f_j^n e^{\otimes n}$  in the local coordinate.

**1.2. Spherical ensemble.** We define the spherical probability measure  $d\nu_n$  to be normalized Haar measure on

$$(12) \quad SH^0(M, L^n) = \{s_n \in H^0(M, L^n) : \|s_n\|_{L^2} = 1\}.$$

We refer to the corresponding probability space as the spherical ensemble. Using the orthonormal basis  $\{s_j^n\}$  we may identify  $SH^0(M, L^n)$  with the unit sphere  $S^{2d_n-1} \subset \mathbb{C}^{d_n}$ .

**1.3. Gaussian measure.** We also endow  $H^0(M, L^n)$  with *normalized Gaussian measures* adapted to the Hermitian metric and the associated inner product (9) on sections as follows: We put

$$(13) \quad d\gamma_n(s_n) = \left(\frac{d_n}{\pi}\right)^{d_n} e^{-d_n|a|^2} da, \quad s_n = \sum_{j=1}^{d_n} a_j^n s_j^n,$$

where  $\{s_1^n, \dots, s_{d_n}^n\}$  is the orthonormal basis of  $H^0(M, L^n)$  with respect to the inner product (9). Equivalently, the coefficients  $a_j^n$  are complex Gaussian random variables which satisfy the following normalization conditions,

$$(14) \quad \mathbb{E}a_k^n = 0, \quad \mathbb{E}a_k^n \bar{a}_j^n = \frac{1}{d_n} \delta_{kj}, \quad \mathbb{E}a_k^n a_j^n = 0$$

Here, we denote the expectation with respect to  $\gamma_n$  by  $\mathbb{E}$ . Under this normalization, we have the expected  $L^2$  norm of  $s_n$ ,

$$(15) \quad \mathbb{E}\|s_n\|_{h^n}^2 = 1.$$

As proved in [FZ], this normalized Gaussian measure is asymptotically equivalent to the spherical measure  $\nu_n$  §1.2 of principal concern in this article.

**1.4. Lift to circle bundle  $X_h$ .** We identify the sections (11) with the locally defined functions

$$(16) \quad s_n = \left(\sum_{j=1}^{d_n} a_j^n f_j^n\right) e^{-\frac{n\phi}{2}}.$$

This identification can be made global by lifting holomorphic sections  $s_n$  of  $L^n$  to equivariant scalar functions  $\hat{s}_n : X_h \rightarrow \mathbb{C}$  on the unit circle bundle  $X_h \rightarrow M$  defined by the metric  $h$  (see [SZ2] for background). That is,

$$(17) \quad X_h = \{v \in L^* : \|v\|_{h^*} = 1\} \rightarrow M$$

where  $\pi : L^* \rightarrow M$  denotes the dual line bundle to  $L$  with dual metric  $h^*$ . We let  $A$  be the connection 1-form on  $X$  given by the Chern  $\nabla$ ; we then have  $dA = \pi^*\omega$ , and thus  $A$  is a contact form on  $X_h$ , i.e.,  $A \wedge (dA)^m$  is a volume form on  $X_h$ .

We let  $r_\theta x = e^{i\theta}x$  ( $x \in X_h$ ) denote the  $S^1$  action on  $X_h$  and denote its infinitesimal generator by  $\frac{\partial}{\partial \theta}$ . A section  $s$  of  $L$  determines an equivariant function  $\hat{s}$  on  $L^*$  by the rule  $\hat{s}(\lambda) = (\lambda, s(z))$  ( $\lambda \in L_z^*, z \in M$ ). We restrict  $\hat{s}$  to  $X_h$  to obtain an equivariant function transforming by  $\hat{s}(r_\theta x) = e^{i\theta}\hat{s}(x)$ . Similarly, a section  $s_n$  of  $L^n$  determines an equivariant function  $\hat{s}_n$  on  $X_h$ : put

$$(18) \quad \hat{s}_n(\lambda) = (\lambda^{\otimes n}, s_n(z)) \quad , \quad \lambda \in X_{h,z} \quad ,$$

where  $\lambda^{\otimes n} = \lambda \otimes \cdots \otimes \lambda$ ; then  $\hat{s}_n(r_\theta x) = e^{in\theta}\hat{s}_n(x)$ . We denote by  $\mathcal{L}_n^2(X_h)$  the space of such equivariant functions transforming by the  $n$ -th character, and by  $\mathcal{H}_n$  the subspace of CR functions annihilated by the tangential Cauchy-Riemann operator  $\bar{\partial}_b$ . We refer to [SZ2] for further details and references.

The space  $\mathcal{H}_n$  carries the natural inner product

$$\langle \hat{s}, \bar{t} \rangle = \int_{X_h} \hat{s} \bar{t} dV_{X_h}, \quad dV_{X_h} = A \wedge (dA)^{m-1}.$$

Lifting the orthonormal basis to  $\{\hat{s}_j^n\}$ , we modify (11) to write every element as

$$\hat{s}_n = \sum_{j=1}^{d_n} a_j^n \hat{s}_j^n.$$

We trivialize the bundle  $X_h \rightarrow M$  (17) using a *Heisenberg coordinate chart* at  $x_0 \in X_h$ , i.e. a coordinate chart of the form

$$(19) \quad \rho(z_1, \dots, z_m, \theta) = e^{i\theta} e^{-\phi} e^*(z),$$

around  $x_0$  where  $(z_1, \dots, z_m)$  are preferred coordinates centered at  $P_0 = \pi(x_0)$  in the sense of [SZ2]. If  $s_n = f e^{\otimes n}$  is a local section of  $L^n$ , then by (18) and (19),

$$(20) \quad \hat{s}_n(z, \theta) = f(z) e^{-n\phi} e^{in\theta}.$$

Thus, (16) is the lift  $\hat{s}_n$  (with the common factor of  $e^{in\theta}$  suppressed).

**1.5. Comparison of  $\gamma_n$  and  $\nu_n$ .** The Gaussian measure  $\gamma_n$  is asymptotically concentrated near the unit sphere as  $d_n \rightarrow \infty$ , so it may be expected that the median and mean of  $\mathcal{L}_\infty^n$  should be asymptotically the same in both ensembles.

The spherical probability measure  $\nu_d$  on the sphere  $S^d(\sqrt{d})$  tends to the Gaussian measure as  $d \rightarrow \infty$  in the following sense: if  $P_d : \mathbb{R}^d \rightarrow \mathbb{R}^k$  is the map,  $P_d(x) = \sqrt{d}(x_1, \dots, x_k)$ , then for all  $k$ ,  $P_{d*}\nu_d \rightarrow \gamma_k = (2\pi)^{-d/2} e^{-|x|^2/2} dx$ . Moreover,

$$(21) \quad \begin{cases} \gamma_d\{x \in \mathbb{R}^d : \|x\|^2 \geq \frac{d}{1-\epsilon}\} \leq e^{-\epsilon^2 d/4}, \\ \gamma_d\{x \in \mathbb{R}^d : \|x\|^2 \leq (1-\epsilon)d\} \leq e^{-\epsilon^2 d/4}. \end{cases}$$

To compare expectations of sup norms, we use the obvious

**LEMMA 1.1.** *The expected values of  $\mathcal{L}_\infty^n$  with respect to the spherical, resp. normalized Gaussian measure, are related by*

$$\mathbb{E}_{\gamma_n} \mathcal{L}_\infty^n = C_n \mathbb{E}_{\nu_n} \mathcal{L}_\infty^n,$$

where

$$C_n = 1 + o(1), \quad n \rightarrow \infty.$$

*Proof.* The one-parameter family of complex Gaussian measures on  $H^0(M, L^n)$  may be written formally as

$$d\gamma_n^\alpha = \left(\frac{\alpha}{\pi}\right)^{d_n} e^{-\alpha\|s\|^2} Ds$$

where  $Ds$  is Lebesgue measure. If we set  $\alpha = d_n$ , then  $d\gamma_n^\alpha$  becomes the normalized Gaussian ensemble  $d\gamma_n$ .

For any  $s \in H^0(M, L^n)$  and for any  $r > 0$ ,  $\mathcal{L}_\infty^n(rs) = \sup_M |rs(z)|_h = r\mathcal{L}_\infty^n(s)$ . Hence,

$$\begin{aligned} \mathbb{E}_{\gamma_n^\alpha} \mathcal{L}_\infty^n &= \frac{\alpha^{d_n}}{\pi^{d_n}} \int_{H^0(M, L^n)} \mathcal{L}_\infty^n(s) e^{-\alpha\|s\|^2} Ds \\ &= \frac{\alpha^{d_n}}{\pi^{d_n}} \omega_{2d_n} \int_0^\infty \int_{SH^0} \mathcal{L}_\infty^n(\tilde{s}) r e^{-\alpha r^2} r^{2d_n-1} dr d\nu_n \\ &= C_n \mathbb{E}_{\nu_n} \mathcal{L}_\infty^n, \end{aligned}$$

where we write  $s = r\tilde{s}$  with  $\tilde{s} \in SH^0(M, L^n)$ , and

$$C_n = \frac{\alpha^{d_n}}{\pi^{d_n}} \omega_{2d_n} \int_0^\infty r e^{-\alpha r^2} r^{2d_n-1} dr = \frac{\alpha^{-\frac{1}{2}}}{2\pi^{d_n}} \omega_{2d_n} \Gamma(d_n + \frac{1}{2}).$$

We then put  $\alpha = d_n$  to obtain

$$C_n = \frac{1}{2d_n^{\frac{1}{2}} \pi^{d_n}} \omega_{2d_n} \Gamma(d_n + \frac{1}{2}).$$

Here,  $\omega_k = \frac{2\pi^{k/2}}{\Gamma(k/2)}$  is the surface measure of the unit sphere  $S^{k-1} \subset \mathbb{R}^k$ . Since  $\frac{\Gamma(d_n + \frac{1}{2})}{\Gamma(d_n)} \sim d_n^{\frac{1}{2}}$  we obtain that  $C_n \simeq 1$ . □

**1.6. Covariance kernel of  $\gamma_n$  and Bergman-Szegö kernels.** The Bergman-Szegö kernel  $\Pi_n(x, y)$  is the Schwartz kernel of the orthogonal projection

$$(22) \quad \Pi_n : \mathcal{L}^2(X_h) \rightarrow \mathcal{H}_n(X_h),$$

i.e.

$$(23) \quad \Pi_n F(x) = \int_{X_h} \Pi_n(x, y) F(y) dV_{X_h}(y), \quad F \in \mathcal{L}^2(X_h).$$

It is given in terms of the orthonormal basis by

$$(24) \quad \Pi_n(x, y) = \sum_{j=1}^{d_n} \hat{s}_j^n(x) \overline{\hat{s}_j^n(y)}.$$

In local coordinates it has the form,

$$(25) \quad \Pi_n(z, w) = \sum_{j=1}^{d_n} f_j^n(z) \overline{f_j^n(w)} e^{-\frac{n(\phi(z) + \phi(w))}{2}}$$

It arises in probability as the covariance kernel of  $\gamma_n$ , i.e.

$$(26) \quad \mathbb{E}(s_n(z) \overline{s_n(w)}) = \frac{1}{d_n} \Pi_n(z, w).$$

The Bergman-Szegö kernels determine Kodaira maps  $\Phi_n : M \rightarrow PH^0(M, L^n)'$  to projective space, defined by [GH]

$$(27) \quad \Phi_n : M \rightarrow \mathbb{CP}^{d_n-1}, \quad \Phi_n(z) = [s_1^n(z) : \cdots : s_{d_n}^n(z)].$$

We lift the maps to  $X_h$  by

$$(28) \quad \tilde{\Phi}_n : X_h \rightarrow \mathbb{C}^{d_n}, \quad \tilde{\Phi}_n(x) = (\hat{s}_1^n(x), \dots, \hat{s}_{d_n}^n(x)).$$

We observe that

$$(29) \quad \Pi_n(x, y) = \tilde{\Phi}_n(x) \cdot \overline{\tilde{\Phi}_n(y)}, \quad \Pi_n(x, x) = \|\tilde{\Phi}_n(x)\|^2.$$

On the diagonal, the Szegő kernel admits a complete asymptotic expansion [Ze],

$$(30) \quad \Pi_n(z, z) = a_0 n^m + a_1(z) n^{m-1} + a_2(z) n^{m-2} + \dots$$

for certain smooth coefficients  $a_j(z)$  with  $a_0 = \pi^{-m}$ . This implies Tian's almost isometry theorem: Let  $\omega_{FS}$  denote the Fubini-Study form on  $\mathbb{CP}^{d_n-1}$ . Then

$$(31) \quad \left\| \frac{1}{n} \Phi_n^*(\omega_{FS}) - \omega \right\|_{\mathcal{C}^k} = O\left(\frac{1}{n}\right)$$

for any  $k$ . We refer to [SZ2] for notation and background. We also need off-diagonal asymptotics of the Bergman kernel [SZ2]. We denote by  $r(z, w)$  the geodesic distance between  $z, w$  with respect to the Kähler metric  $\omega$  on  $M$ .

**THEOREM 1.2.** a) *Within a  $\frac{C}{\sqrt{n}}$  neighborhood of the diagonal, the Bergman-Szegő kernel is given by the scaling asymptotics:*

$$(32) \quad n^{-m} \Pi_N(z_0 + u/\sqrt{n}, \theta/n; z_0 + v/\sqrt{n}, 0) \sim \Pi_1^H(u, \theta; v, 0) [1 + O(1/\sqrt{n})].$$

Here

$$\Pi_1^H(u, \theta; v, \psi) = \frac{1}{\pi^m} e^{i(\theta-\psi) + i\Im(u \cdot \bar{v}) - \frac{1}{2}|u-v|^2}$$

is the Szegő kernel of the reduced Heisenberg group.

b) *If  $r(z, w) \leq C/n^{1/3}$ , we have:*

$$(33) \quad |\Pi_n(z, w)| \leq \left( \frac{1}{\pi^m} + o(1) \right) n^m \exp \left( -\frac{1-\varepsilon}{2} n d(z, w)^2 \right) + O(n^{-\infty}).$$

c) *On all of  $M$ , we have:*

$$(34) \quad |\Pi_n(z, w)| \leq C n^m \exp(-\lambda \sqrt{n} d(z, w)).$$

for some positive  $\lambda > 0$ .

The estimate (b) on the larger  $n^{-1/3}$  balls is from [SZ2, Lemma 5.2(ii)]. The off-diagonal estimate (c) follows by an Agmon distance argument, as noted by M. Christ [Ch1].

## 2. THE METRICS $\mathbf{d}_n$

The proof of our main result is based on the Dudley's metric entropy method which relates the median of the suprema of a process by its 'pseudometric'. In this section, we will compute the pseudometric (which turns out to be a metric) for our normalized Gaussian ensemble and the spherical ensemble. There is a clash of notation between the dimension  $d_n$  of  $H^0(M, L^n)$  and the metric  $\mathbf{d}_n$ , but both are standard and we distinguish them by putting the metric in boldface. Recall that  $r(z, w)$  denotes the geodesic distance between  $z, w$  with respect to the Kähler metric  $\omega$  on  $M$ .



LEMMA 2.1. *In the the spherical  $\nu_n$  and normalized Gaussian ensemble  $\gamma_n$ , we have*

$$(35) \quad \mathbf{d}_n(z, w) = \frac{1}{\sqrt{d_n}} \sqrt{\Pi_n(z, z) + \Pi_n(w, w) - 2\Re \Pi_n(z, w)}$$

where  $\Pi_n(z, w)$  is the Szegő kernel in (25).

*Proof.* We first consider  $\gamma_n$ , the normalized Gaussian random sections (13) (or the equivalent expression (16)). By definition and by (14),

$$\begin{aligned} \mathbf{d}_n^2(z, w) &= \mathbb{E} \left| \left( \sum_{j=1}^{d_n} a_j^n f_j^n(z) \right) e^{-\frac{n\phi(z)}{2}} - \left( \sum_{j=1}^{d_n} a_j^n f_j^n(w) \right) e^{-\frac{n\phi(w)}{2}} \right|^2 \\ &= \mathbb{E} \left( \sum_{j,k=1}^{d_n} a_j^n \bar{a}_k^n f_j^n(z) \overline{f_k^n(z)} e^{-n\phi(z)} + \sum_{j,k=1}^{d_n} a_j^n \bar{a}_k^n f_j^n(w) \overline{f_k^n(w)} e^{-n\phi(w)} \right. \\ &\quad \left. - 2\Re \mathbb{E} \left( \sum_{j,k=1}^{d_n} a_j^n \bar{a}_k^n f_j^n(z) \overline{f_k^n(w)} \right) e^{-\frac{n\phi(z)}{2} - \frac{n\phi(w)}{2}} \right) \\ &= \frac{1}{d_n} (\Pi_n(z, z) + \Pi_n(w, w) - 2\Re \Pi_n(z, w)). \end{aligned}$$

We then observe that the expectations for  $\nu_n$  are the same as in (14):

$$\nu_n(a_j \bar{a}_k) = \begin{cases} 0, & k \neq j \\ \frac{1}{d_n}, & k = j. \end{cases}$$

Indeed, for  $j \neq k$ ,  $a_j \bar{a}_k$  is a homogeneous harmonic polynomial of degree 2 on  $\mathbb{C}^{d_n} \simeq \mathbb{R}^{2d_n}$ . Indeed, if we write  $a_j = u_j + iv_j$  (and similarly for  $a_k$  then  $a_j \bar{a}_k = (u_j u_k + v_j v_k) + i(u_j v_k - u_k v_j)$  and  $\Delta_{\mathbb{R}^{2n}} = \Delta_u + \Delta_v$  will annihilate it. Hence its restriction to  $S^{2d_n-1}$  is a spherical harmonic of degree 2 and it is orthogonal to the constant function, proving the first statement. For the second we use that  $\mathbb{E}|a_j|^2$  is independent of  $j$  and therefore equals its average. It follows that  $\mathbf{d}_{\nu_n} = \mathbf{d}_n$ .  $\square$

We may interpret the distance in terms of the lifted Kodaira embeddings (28).

LEMMA 2.2.  $\mathbf{d}_n(z, w) = \frac{1}{\sqrt{d_n}} \|\tilde{\Phi}_n^z - \tilde{\Phi}_n^w\|_{L^2}$ . Thus  $\mathbf{d}_n(z, w)$  is a metric on the Kähler manifolds.

*Proof.* By (29) have,

$$\|\tilde{\Phi}_n^z - \tilde{\Phi}_n^w\|^2 = \|\tilde{\Phi}_n^z\|^2 + \|\tilde{\Phi}_n^w\|^2 - 2\Re \langle \tilde{\Phi}_n^z, \tilde{\Phi}_n^w \rangle.$$

$d_n(z, w)$  is a metric since it satisfies the triangle inequality which is equivalent to the Minkowski inequality of the  $L^2$ -norm  $\|\cdot\|_{L^2}$ .  $\square$

Since  $d_n$  is asymptotically of order  $n^m$  by (8),  $\mathbf{d}_n(z, w)$  is roughly  $n^{-m/2}$  times the distance in  $\mathbb{C}^{d_n}$  between  $\tilde{\Phi}_n^z$  and  $\tilde{\Phi}_n^w$ . The distance  $\mathbf{d}_n$  is globally very different from the Riemannian distance on  $M$  defined by the Kähler metric  $\omega$ . However by (31), the Kodaira embeddings are almost isometric on tangent planes, hence for distances of order  $n^{-\frac{1}{2}}$  they nearly isometric. This is the key idea needed to calculate the covering numbers  $N(M, \mathbf{d}_n, \epsilon)$ , and then the metric entropies asymptotically.

The next Lemma gives the asymptotics of  $\mathbf{d}_n$  for separated  $(z, w)$ :

LEMMA 2.3. *For all  $z, w$ ,*

$$(36) \quad \mathbf{d}_n(z, w) \leq \sqrt{2}.$$

*Moreover, for  $z, w$  with  $r(z, w) > \frac{c \log n}{\sqrt{n}}$  in the geodesic distance of  $(M, \omega)$ , then*

$$(37) \quad \mathbf{d}_n(z, w) = \sqrt{\frac{2}{d_n}(\Pi_n(z, z) + \Pi_n(w, w))} + O(e^{-\sqrt{n}|z-w|}) \simeq \sqrt{2} + O\left(\frac{1}{\sqrt{n}}\right) + O(e^{-\sqrt{n}|z-w|})$$

*for sufficiently large  $n$ .*

*Remark:* The second statement can be interpreted as follows:  $n^{-m/2}\tilde{\Phi}_n^z$  is almost a unit vector, and  $n^{-m/2}\tilde{\Phi}_n^z$  is almost orthogonal to  $n^{-m/2}\tilde{\Phi}_n^w$  if  $r(z, w) \geq \frac{c \log n}{\sqrt{n}}$ .

*Proof.* By Lemma 2.2, we have

$$\mathbf{d}_n(z, w) = \frac{1}{d_n} \|\tilde{\Phi}_n^z - \tilde{\Phi}_n^w\|_{L^2} \leq \frac{1}{d_n} (\|\tilde{\Phi}_n^z\|_{L^2} + \|\tilde{\Phi}_n^w\|_{L^2})$$

The first inequality follows from the asymptotics of  $d_n$  (8) and  $\|\tilde{\Phi}_n^z\|$  (29)(30). The inequality (37) is the consequence of Theorem 1.2.  $\square$

We then have the asymptotics of the distance between very close points:

LEMMA 2.4. *We have asymptotics,*

$$(38) \quad \mathbf{d}_n(z, w) \simeq \begin{cases} \sqrt{1 - e^{-nr^2(z, w)}}, & r(z, w) < cn^{-\frac{1}{2}} \log n \\ n^{\frac{1}{2}} r(z, w), & r(z, w) < cn^{-\frac{1}{2} - \eta}. \end{cases}$$

*for any  $\eta > 0$ . Equivalently,*

$$(39) \quad \mathbf{d}_n\left(z + \frac{u}{\sqrt{n}}, z + \frac{v}{\sqrt{n}}\right) \sim |u - v|.$$

*Here,  $|u - v|$  is the Euclidean distance in normal coordinates.*

*Proof.* By Theorem 1.2 with  $r(z, w) \leq \frac{c \log n}{\sqrt{n}}$ ,

$$\Pi_n(z, w) \sim e^{n(\phi(z, w) - \frac{1}{2}\phi(z) - \frac{1}{2}\phi(w))} A_n(z, w)$$

where

$$A_n = n^m \left(1 + \frac{1}{n} a_1 + \dots\right)$$

and where  $\phi(z, w)$  is the almost analytic extension of the Kähler potential  $\phi(z)$  (see [SZ2] for background). In particular, if  $r(z, w) < cn^{-\frac{1}{2} - \eta}$ , it follows from Theorem 1.2 that

$$\begin{aligned} \mathbf{d}_n(z, w) &= \frac{1}{\sqrt{d_n}} \sqrt{\Pi_n(z, z) + \Pi_n(w, w) - 2\Re \Pi_n(z, w)} \\ &\sim \sqrt{1 - e^{-nr^2(z, w)}} \sim n^{\frac{1}{2}} r(z, w). \end{aligned}$$

$\square$

**Example:**  $(\mathbb{CP}^m, \omega_{FS})$  In the case of the  $SU(m+1)$  ensemble where  $M = \mathbb{CP}^m$  and  $\omega = \omega_{FS}$ , the Fubini-Study metric, the lifted Szegő kernel on  $S^{2m-1}$  is

$$(40) \quad \Pi_n(x, y) = \frac{(n+m)!}{\pi^m n!} \langle x, \bar{y} \rangle^n.$$

It is constant on the diagonal, equal to  $\frac{1}{\text{Vol}(\mathbb{CP}^m)}$  times the dimension  $d_n = \dim H^0(\mathbb{CP}^m, \mathcal{O}(n))$ . The lifted distance on  $X_h = S^{2m-1}$  is then,

$$\mathbf{d}_n(x, y) = \frac{1}{\sqrt{\text{Vol}(\mathbb{CP}^m)}} \sqrt{2 - 2\Re \langle x, \bar{y} \rangle^n} = \frac{\sqrt{2}}{\sqrt{\text{Vol}(\mathbb{CP}^m)}} \sqrt{1 - \cos^n r(z, w)}.$$

where the last equation holds when  $x = (z, 0), y = (w, 0)$  (i.e. the angle in  $S^1$  of the projection  $S^1 \rightarrow S^{2m-1} \rightarrow \mathbb{CP}^m$  is zero in local coordinates). Since  $\cos r = 1 - \frac{r^2}{2} + O(r^4)$ ,

$$\cos^n r = e^{n \log(1 - \frac{r^2}{2} + O(r^4))} = e^{-n(\frac{r^2}{2} + O(r^4))} = e^{-n\frac{r^2}{2}}(1 + O(nr^4)),$$

so the remainder term is negligible as long as  $r \leq Cn^{-\frac{1}{4}-\epsilon}$ . In this range,

$$\mathbf{d}_n(z, w) \simeq \frac{\sqrt{2}}{\sqrt{\text{Vol}(\mathbb{CP}^m)}} \sqrt{1 - e^{-n\frac{r^2(z, w)}{2}}}.$$

Summarizing Lemmas 2.3 - 2.4 (and assuming  $\text{Vol}_\omega(M) = 1$ ),

$$\mathbf{d}_n \sim \sqrt{2} \sqrt{1 - e^{-nr^2/2}}, \quad r \leq c \frac{\log n}{\sqrt{n}}$$

or equivalently,

$$(41) \quad r^2(z, w) \sim \frac{2}{n} \log(1 - \frac{1}{2} \mathbf{d}_n^2)^{-1}, \quad \mathbf{d}_n \in [0, \sqrt{2} \sqrt{1 - e^{-\frac{c^2}{2}(\log n)^2}}]$$

We denote  $N(M, \omega, \epsilon')$  as the number of geodesic balls of radius  $\epsilon'$  to cover the Kähler manifolds. Then we have relation,

**COROLLARY 2.5.** *The covering number  $N(M, \mathbf{d}_n, \epsilon)$  satisfies:*

$$(42) \quad N(M, \mathbf{d}_n, \epsilon) = \begin{cases} N(M, \omega, \sqrt{\frac{2}{n}} \sqrt{\log(1 - \frac{1}{2} \epsilon^2)^{-1}}), & \epsilon \leq \sqrt{2} \sqrt{1 - \frac{1}{n}}, \\ [1, N(M, \omega, \sqrt{\frac{2 \log n}{n}})], & \epsilon \geq \sqrt{2} \sqrt{1 - \frac{1}{n}} \end{cases}$$

*Proof.* The first line is the direct consequence of the formula (41), the only thing we need to check is  $\sqrt{2} \sqrt{1 - \frac{1}{n}} \in [0, \sqrt{2} \sqrt{1 - e^{-\frac{c^2}{2}(\log n)^2}}]$ , this is true as  $n$  large enough.

For the second line, we know in Lemma 2.2 that  $\mathbf{d}_n$  is a metric, thus  $N(M, \mathbf{d}_n, \epsilon)$  will be a decreasing function with respect to the radius  $\mathbf{d}_n = \epsilon$ , thus  $N(M, \mathbf{d}_n, \epsilon)$  will be bounded by the ends points  $\epsilon = \infty$  and  $\sqrt{2} \sqrt{1 - \frac{1}{n}}$ . For  $\epsilon = \sqrt{2} \sqrt{1 - \frac{1}{n}}$ , the number of balls we need to covering the manifold will be  $N(M, \omega, \sqrt{\frac{2 \log n}{n}})$  by formula (41); for  $\epsilon = \infty$ , we know the diameter of the manifolds is  $\mathbf{d}_n(z, w) \leq \sqrt{2}$  for all  $z, w \in M$  in Lemma 2.3, thus we only need 1 ball to cover the manifold if  $\epsilon \geq \sqrt{2}$ . □

## 3. UPPER BOUND AND LOWER BOUND

In this section we prove Theorem 1 except that we do not give sharp estimates on the coefficients of  $\sqrt{\log n}$ . They are proved in the last section.

**3.1. Bound for mean.** The following extends the non-sharp version of Theorem 1 to the normalized Gaussian ensemble as well as the spherical ensemble:

**THEOREM 3.1.** *Let  $\mathbb{E}_n$  denote the expectation with respect to either the spherical ensemble  $\nu_n$  or the normalized Gaussian ensemble  $\gamma_n$ . Then we have bounds,*

$$(43) \quad c\sqrt{\log n} \leq \mathbb{E}_n \mathcal{L}_\infty^n \leq C\sqrt{\log n}.$$

where  $c$  and  $C$  can be chosen to be the same in both ensembles by Lemma 1.1.

**3.1.1. Upper bound.** The upper will be given by the Dudley entropy bound of Theorem 2, which holds for the  $\gamma_n$  and for  $\nu_n$  since the latter is sub-Gaussian. Since the metrics  $\mathbf{d}_n$  are the same in both ensembles, the same upper bound will hold.

By Lemma 2.3,  $N(M, \mathbf{d}_n, \epsilon) = 1$  if  $\epsilon > \sqrt{2}$ , i.e.,  $\log N(M, \mathbf{d}_n, \epsilon) = 0$ . Hence,

$$(44) \quad \mathbb{E}_n \sup_M |s_n|_{h^n} \leq C \int_0^{\sqrt{2}} \sqrt{\log N(M, \mathbf{d}_n, \epsilon)} d\epsilon.$$

We break up the integral (44) into two terms,

$$\int_0^{\sqrt{2}} = \int_0^{\sqrt{2}\sqrt{1-\frac{1}{n}}} + \int_{\sqrt{2}\sqrt{1-\frac{1}{n}}}^{\sqrt{2}} := I_n + II_n.$$

In integral  $I$ , it follows from Corollary 2.5, we have (for a constant  $C_m > 0$  which depends only on the dimension  $m = \dim_{\mathbb{C}} M$  but which changes line to line),

$$(45) \quad \begin{aligned} I_n &\leq \int_0^{\sqrt{2}\sqrt{1-\frac{1}{n}}} \sqrt{\log N\left(M, \omega, \sqrt{\frac{2}{n}}\sqrt{\log(1-\frac{1}{2}\epsilon^2)^{-1}}\right)} d\epsilon \\ &\leq C_m \int_0^{\sqrt{2}\sqrt{1-\frac{1}{n}}} \sqrt{\log\left(\sqrt{\frac{2}{n}}\sqrt{\log(1-\frac{1}{2}\epsilon^2)^{-1}}\right)^{-2m}} d\epsilon \\ &\leq C_m \sqrt{\log \frac{n}{2}} \int_0^{\sqrt{2}\sqrt{1-\frac{1}{n}}} \sqrt{\left(1 - \frac{2}{\log \frac{n}{2}} \log(\sqrt{\log(1-\frac{\epsilon^2}{2})^{-1}})\right)} d\epsilon \\ &= C_m \sqrt{\log \frac{n}{2}} \int_0^{\sqrt{2}\sqrt{1-\frac{1}{n}}} \sqrt{\left(1 - \frac{1}{\log \frac{n}{2}} \log \log(1 - \frac{\epsilon^2}{2})^{-1}\right)} d\epsilon. \end{aligned}$$

By dominated convergence,

$$\int_0^{\sqrt{2}\sqrt{1-\frac{1}{n}}} \sqrt{\left(1 - \frac{1}{\log \frac{n}{2}} \log \log(1 - \frac{\epsilon^2}{2})^{-1}\right)} d\epsilon \rightarrow \sqrt{2}.$$

Hence,

$$(46) \quad I_n = C_m \sqrt{\log n} (1 + o(1)).$$

where  $C_m$  depends only on the dimension.

On the other hand, by the second part of Corollary 2.5,

$$(47) \quad II_n \leq \sqrt{2 \log N(M, \omega, \sqrt{\frac{2 \log n}{n}})} \left(1 - \sqrt{1 - \frac{1}{n}}\right) \ll \sqrt{\log n}.$$

Combining (46)- (47) completes the proof.

**3.1.2. Lower bound.** For the normalized Gaussian ensemble, the lower bound is given by the Sudakov minoration principle,

$$(48) \quad \mathbb{E}_{\gamma_n} \sup_M |s_n| \geq c_m \epsilon \sqrt{\log N(M, \mathbf{d}_n, \epsilon)}, \text{ for all } \epsilon > 0.$$

To obtain the lower bound it suffices to choose an optimal value of  $\epsilon$ . As in the calculation of Dudley's integral, for  $\epsilon \in [0, \sqrt{2}\sqrt{1 - \frac{1}{n}}]$ , we have

$$\epsilon \sqrt{\log N(M, \mathbf{d}_n, \epsilon)} \sim \epsilon \sqrt{\log \frac{n}{2}} \sqrt{\left(1 - \frac{1}{\log \frac{n}{2}} \log \log(1 - \frac{\epsilon^2}{2})^{-1}\right)} \geq c \sqrt{\log n}$$

if we choose some  $\sqrt{2} > b > \epsilon = a > 0$  for  $n$  large enough.

The lower bound for the spherical ensemble then follows by Lemma 1.1.

**3.2. Bounds for median.** We now prove the non-sharp bounds for the median.

**THEOREM 3.2.** *We have the following bounds for the median under the spherical ensemble  $(SH^0(M, L^n), \nu_n)$ ,*

$$(49) \quad c \sqrt{\log n} \leq \mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n) \leq C \sqrt{\log n}.$$

where the constants  $c$  and  $C$  can be chosen to be the same as the ones in (43).

The proof of Theorem 3.2 and Corollary 1 are based on the well-known Levy concentration of measure theorem [Le] for Lipschitz continuous functions on spheres of large dimension. Let  $\mathcal{M}(f)$  denote the median of  $f$ . Then

$$(50) \quad \mathbb{P} \{x \in S^d : |f(x) - \mathcal{M}(f)| \geq r\} \leq \exp \left( -\frac{(d-1)r^2}{2\|f\|_{Lip}^2} \right),$$

where

$$\|f\|_{Lip} = \sup_{d(x,y)>0} \frac{|f(x) - f(y)|}{d(x,y)}$$

is the Lipschitz norm.

We apply this result to  $f = \mathcal{L}_\infty^n$ . In [SZ] it is observed that

- (i)  $\mathcal{L}_\infty^n$  is Lipschitz continuous with norm  $\frac{n^{m/2}}{\sqrt{\log n}} \leq \|\mathcal{L}_\infty^n\|_{Lip} \leq n^{m/2}$ .
- (ii) The median of  $\mathcal{L}_\infty^n$  satisfies:  $\mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n) \leq C_m \sqrt{\log n}$  for sufficiently large  $n$ .

The estimate of the Lipschitz norm, is based on the fact that the  $\mathcal{L}^2$ -normalized 'coherent states'  $\Phi_n^w(z) = \frac{\Pi_n(z,w)}{\sqrt{\Pi_n(w,w)}}$  are the global maxima of  $\mathcal{L}_n^\infty$  on  $SH^0(M, L^n)$  and that  $\|\Phi_n^w(z)\|_\infty = \sqrt{\Pi_n(w,w)} \sim n^{m/2}$ . It follows that

$$||s_1 + s_2||_\infty - ||s_1||_\infty| \leq 3n^{m/2}.$$

Now let  $s_1$  have  $L^\infty$  norm  $\leq C\sqrt{\log n}$  and let  $s_1 = \Phi_n^w$  for some  $w$ . Then

$$\left| \|s_1 + s_2\|_\infty - \|s_1\|_\infty \right| \geq \frac{n^{m/2}}{\sqrt{\log n}}.$$

Summarizing the facts in our setting, we can rewrite (50) as

$$(51) \quad \mathbb{P}(|\mathcal{L}_\infty^n - \mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n)| > r) \leq e^{-\frac{r^2}{2}}$$

for  $n$  large enough. Then the difference of the mean and median of sup norm is estimated to be,

$$\begin{aligned} |\mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n) - \mathbb{E}_{\nu_n}(\mathcal{L}_\infty^n)| &\leq \mathbb{E}_{\nu_n} |\mathcal{L}_\infty^n - \mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n)| \\ &= \int_0^\infty \mathbb{P}(|\mathcal{L}_\infty^n - \mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n)| > a) da \\ &= \int_0^{c\sqrt{\log n}} + \int_{c\sqrt{\log n}}^\infty \\ &\leq c\sqrt{\log n} + \int_{c\sqrt{\log n}}^\infty e^{-\frac{a^2}{2}} da \\ &\leq c\sqrt{\log n} + n^{-\frac{c^2}{2}} \end{aligned}$$

for any positive constant  $c > 0$ . This implies that the difference of median and mean is bounded by  $c\sqrt{\log n}$  for small  $c > 0$ .

**COROLLARY 3.3.** *It follows that*

$$\frac{1}{\sqrt{\log n}} (\mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n) - \mathbb{E}_{\nu_n}(\mathcal{L}_\infty^n)) \rightarrow 0$$

as  $n \rightarrow \infty$ .

Thus Theorem 3.2 follows from Corollary 3.3 together with the bounds in Theorem 3.1.

#### 4. SHARP BOUNDS

In this section, we prove that the coefficients  $c$  and  $C$  in the upper and lower bounds of Theorem 3.1 and Theorem 3.2 in Section 3 can be taken to be  $\sqrt{m}$ .

**4.1. Upper bounds using the value density.** We first prove the sharp upper bound in Theorem 1 using the results of [FZ] on the expected distribution of critical values of the spherical random sections.

We define the normalized empirical measure of the critical values of random sections by,

$$(52) \quad CV_n(s_n) = \frac{1}{n^m} \sum_{z: \nabla_n s_n = 0} \delta_{|s_n|_{h^n}}$$

where  $s_n \in SH^0(M, L^n)$  and  $\nabla_n$  is the Chern connection of the line bundle  $L^n$  with respect to the Hermitian metric  $h^n$  [GH]. The expected density of critical values for the spherical ensemble was determined in [FZ] to be given by

$$(53) \quad \lim_{n \rightarrow \infty} \mathbb{E}_{\nu_n} CV_n = p(x) e^{-x^2}$$

in the sense of distribution, where  $p(x)$  is a smooth function with polynomial growth (Theorem 1 and Theorem 2 in [FZ]). In fact,  $p(x) \sim c_m x^{m(m+1)+1}$  where  $c_m$  is a universal constant only depending on the dimension and independent of  $n$ .

We define the random variable,

$$(54) \quad X_a = \langle \sum_{\nabla_n s_n = 0} \delta_{|s_n|_{h^n}}, 1_{[a, \infty)} \rangle$$

which is the number of critical values that fall in the interval  $[a, \infty)$ . Then,

$$(55) \quad \mathbb{P}(\sup_M |s_n|_{h^n} \geq a) = \mathbb{P}(X_a \geq 1)$$

By Chebyshev's inequality, we have,

$$(56) \quad \mathbb{P}(X_a \geq 1) \leq \mathbb{E}X_a \sim n^m \int_a^\infty p(x) e^{-x^2} dx$$

for  $n$  large enough.

Recall for any nonnegative random variable  $X$ , we have the identity,

$$(57) \quad \mathbb{E}X = \int_0^\infty \mathbb{P}(X > a) da$$

Letting  $X \sup_M |s_n|_{h^n}$  gives

$$(58) \quad \mathbb{E}_{\nu_n} \sup_M |s_n|_{h^n} = \int_0^\infty \mathbb{P}(\sup_M |s_n|_{h^n} > a) da = \int_0^{c\sqrt{\log n}} + \int_{c\sqrt{\log n}}^\infty =: I + II$$

The first term is bounded by  $c\sqrt{\log n}$  since the probability is always less than 1. For the second term, we apply formulas (55)(56), by choosing suitable constant  $c$ , as  $n$  large enough we will have,

$$II \leq n^m \int_{c\sqrt{\log n}}^\infty \int_a^\infty p(x) e^{-x^2} dx da \sim n^m \int_{c\sqrt{\log n}}^\infty p_1(a) e^{-a^2} da \leq Cn^{-k}$$

for some constant  $C > 0$  and  $k > 0$ , where in the second inequality we use the integration by part several times and  $p_1(a)$  is a smooth function with polynomial growth. The upper bound in Theorem 3.1 follows from these estimates of  $I$  and  $II$ . To obtain the optimal  $C = C_m$  depending only on the dimension we consider the minimum value  $C_m$  of  $C$  so that

$$\mathbb{P}(\sup_M |s_n|_{h^n} > C\sqrt{\log n}) \leq \frac{1}{2}.$$

Setting  $a = C\sqrt{\log n}$  in (56), we get for sufficiently large  $n$ ,

$$\begin{aligned} \mathbb{P}(\sup_M |s_n|_{h^n} > C\sqrt{\log n}) &\leq cn^m \int_{C\sqrt{\log n}}^\infty x^{m(m+1)+1} e^{-x^2} dx \\ &\leq cn^m (C\sqrt{\log n})^{m(m+1)} e^{-C^2 \log n} \\ &\leq \frac{1}{2}, \quad \text{as long as } C \geq \sqrt{m + \frac{m(m+1)}{2} \frac{\log \log n}{\log n}}. \end{aligned}$$

It follows that

$$\limsup_{n \rightarrow \infty} \frac{\mathcal{M}_{\nu_n}(\mathcal{L}_\infty^n)}{\sqrt{\log n}} \leq \sqrt{m}$$

and by Corollary 3.3, we have,

$$(59) \quad \limsup_{n \rightarrow \infty} \frac{\mathbb{E}_{\nu_n} \mathcal{L}_{\infty}^n}{\sqrt{\log n}} \leq \sqrt{m}.$$

**4.2. Lower bound.** The lower bound of Theorem 1 for the mean follows from a precise analysis of the constant in Sudakov's minoration (48) for the normalized Gaussian case.

There is a universal estimate of the constant appearing in (48). We follow [Li], Lemma 10.2. In Lemma 10.2, the numerical constant  $c_* = 0.64$  is chosen such that the inequality (10.8) is true for any integer  $n$ , but as stated in the proof of Lemma 10.2, the constant  $c$  can be chosen to be any  $c < \sqrt{2}$  if we have infinite many points. If we combine this with Theorem 10.5 in [Li], we have

$$\mathbb{E} \sup_T X \geq \epsilon \sqrt{\log N(T, \epsilon)}$$

in Sudakov's minoration.

Our Gaussian random fields are complex valued and are therefore equivalent to a real two dimensional Gaussian process. Write  $X = Y + iZ$  for two real standard independent Gaussian processes. Then we have,

$$\sup |X| = \sup \sqrt{Y^2 + Z^2} \geq \frac{1}{\sqrt{2}} \sup |Y + Z| \geq \frac{1}{\sqrt{2}} \sup (Y + Z)$$

Then we apply Sudakov's minoration to the real process  $Y + Z$  to get the lower bound,

$$\mathbb{E} \sup |X| \geq \frac{\epsilon}{\sqrt{2}} \sqrt{\log N(T, \epsilon)}$$

where the  $L^2$  metric is given by  $d(t, s) = \sqrt{\mathbb{E}(Y_t + Z_t - Y_s - Z_s)^2} = \sqrt{\mathbb{E}|X_t - X_s|^2}$ .

In our case, we will have,

$$(60) \quad \mathbb{E}_{\gamma_n} (\sup_M |s_n|_{h^n}) \geq \frac{\epsilon}{\sqrt{2}} \sqrt{\log N(M, \mathbf{d}_n, \epsilon)}$$

for any  $\epsilon > 0$ . In the proof of the lower bound we found that for  $\epsilon = \sqrt{2} \sqrt{1 - \frac{1}{n}}$ ,  $N(M, \mathbf{d}_n, \epsilon) = N(M, \omega, \sqrt{\frac{2 \log n}{n}}) = (\frac{2 \log n}{n})^{-m}$  (Corollary 2.5), so by (60) we have

$$\mathbb{E}_{\gamma_n} \sup_M |s_n|_{h^n} \geq \sqrt{m \log n}$$

for  $n$  large enough. It follows that, for the normalized Gaussian sections in Theorem 3.1, we have

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}_{\gamma_n} \mathcal{L}_{\infty}^n}{\sqrt{\log n}} \geq \sqrt{m}$$

The lower bounds for the spherical measures  $\nu_n$  then follow from Lemma 1.1,

$$(61) \quad \liminf_{n \rightarrow \infty} \frac{\mathbb{E}_{\nu_n} \mathcal{L}_{\infty}^n}{\sqrt{\log n}} \geq \sqrt{m}$$

Thus we have the sharp estimate of the mean in Theorem 1 if we combine (59)(61). The sharp estimate for the median in Theorem 1 follows from Corollary 3.3.



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